Quantum Contextuality within the Quantum Theory Itself, without a Reference to Any Classical Theory

Mladen Pavičić

Institute Ruđer Bošković (IRB), Zagreb, Croatia.



Quantum Information and Probability: from Foundations to Engineering (QIP26) conference,

June 9–12, 2026 — Linnaeus University, Växjö, Sweden

QIP26

Hypergraph notation; Vertex; Hyeperedge

hypergraph = pair v-h;

v = a set of elements called vertices;

h = a set of non-empty subsets of v called hyperedges;

hyperedge = a set of vertices *related* to each other — e.g., *orthogonal* to each other.

Each vertex is denoted by one of the following characters:

`1 2 ... 9 A ... Z a ... z ! ... ~ +1 ... ++1 ...`

McKay-Megill-Pavičić hypergraph (MMPH) language

Dimensionality n means a maximal number of vertices (n) within a hyperedge.

An MMPH is an n-dim hypergraph in which

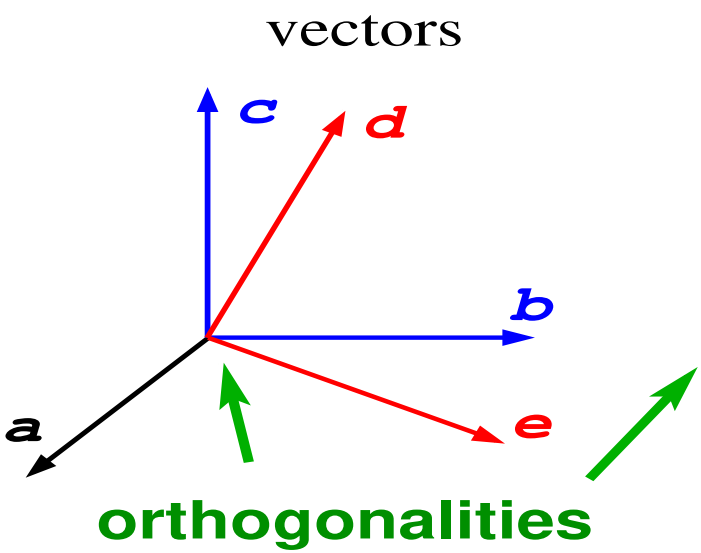
(i) Every vertex belongs to at least one hyperedge.

(ii) Every hyperedge contains at least 2 vertices;

(iii) Hyperedges may intersect each other in at most $n-2$ vertices

(iv) No hyperedge shares only one vertex with another hyperedege;

McKay-Megill-Pavičić hypergraph (MMPH) strings

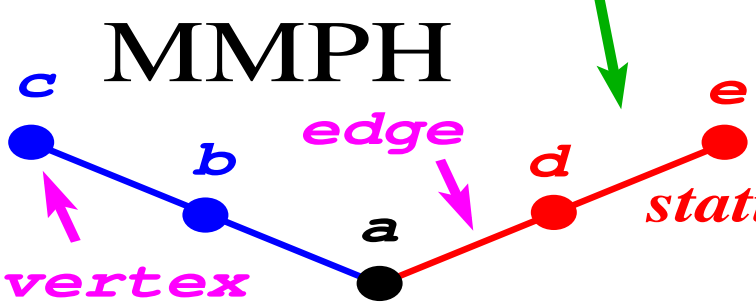


system of nonlinear equations

$$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z = 0$$
$$\mathbf{a}\cdot\mathbf{c} = a_x c_x + a_y c_y + a_z c_z = 0$$
$$\mathbf{b}\cdot\mathbf{c} = b_x c_x + b_y c_y + b_z c_z = 0$$
$$\mathbf{a}\cdot\mathbf{d} = a_x d_x + a_y d_y + a_z d_z = 0$$
$$\mathbf{a}\cdot\mathbf{e} = a_x e_x + a_y e_y + a_z e_z = 0$$
$$\mathbf{d}\cdot\mathbf{e} = d_x e_x + d_y e_y + d_z e_z = 0$$

coordinatization

exponential complexity



statistically polynomial complexity

MMPH string: **cba,ade.**

Contextuality vs. noncontextuality

n-dim MMPH *non-binary* contextual set, $n \geq 3$, is a hypergraph whose each hyperedge contains at least two and at most *n* vertices to which it is impossible to assign 1s and 0s in such a way that

(i) No two vertices within any of its hyperedges are both assigned 1;

(ii) In any of its hyperedges, not all of the vertices are assigned 0.

****

A *binary* MMPH set to which it is possible to assign 1s and 0s so as to satisfy the above two conditions is noncontextual.

****

N.B. No coordinatization is needed. Vertices need not be vectors.

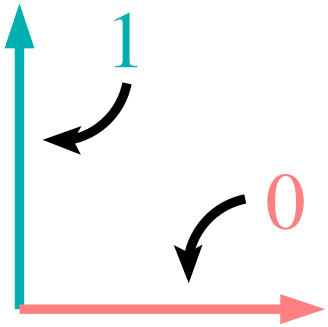
****

MMPH non-binary conditions visualised — for vectors

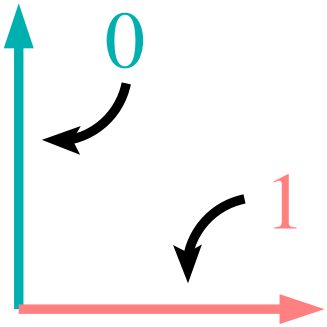
MMPH non–binary set violates the following conditions:

(*i*)

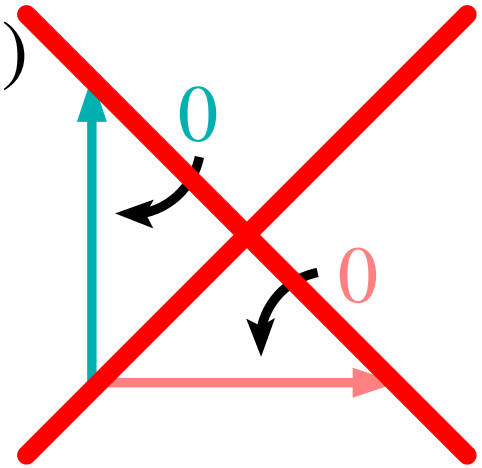
either



or



(*ii*)



KS vs. non-KS non-binary MMPHs

A non-binary contextual n-dim MMPH with all hyperedges containing n vertices, **with a coordinatization**, is called a KS set or a Kochen-Specker MMPH.

A non-binary contextual n-dim MMPH with at least one of the hyperedges containing less than n vertices, **with a coordinatization**, is called a non-KS MMPH or a non-Kochen-Specker MMPH.

MMPH contextuality, criticality, and smallness

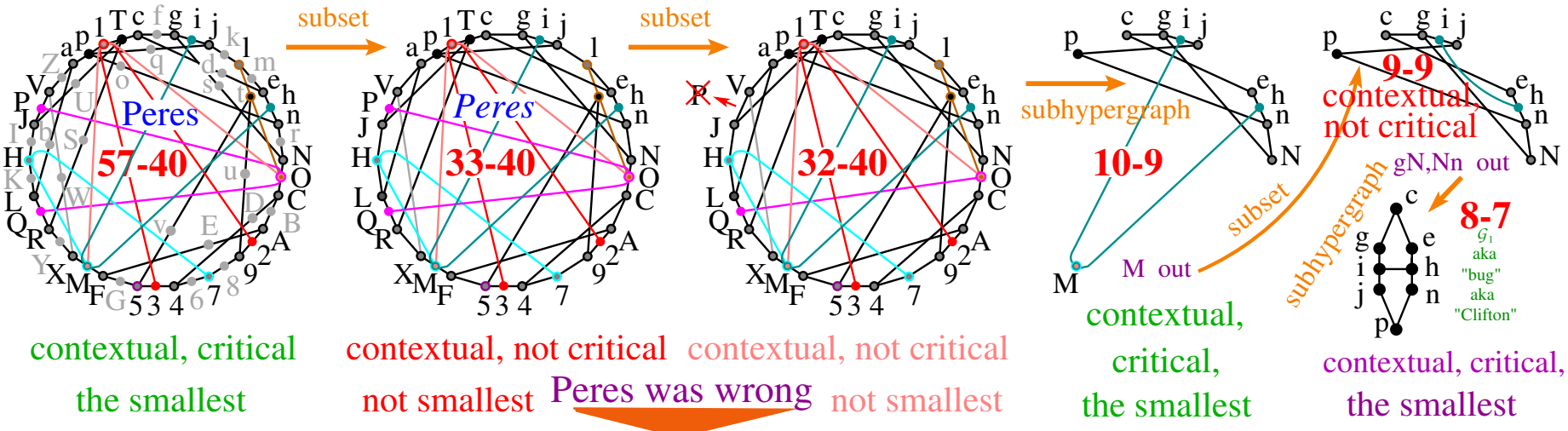
MMPH master: an MMPH which contains a number of smaller MMPHs.

Critical MMPH: dropping any of the hyperedges renders it non-contextual.

The smallest critical MMPH is the smallest contextual subhypergraph of its master.

Peres' 57-40 is the smallest critical subhypergraph of its master 81-52.

Peres' 33-40 is not critical and is not a subhypergraph of 57-40.



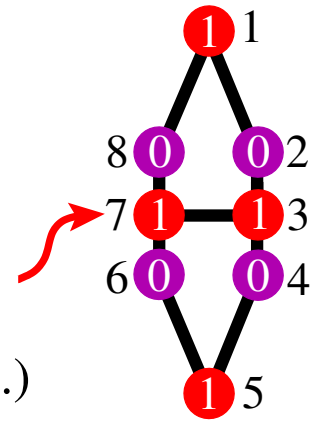
"It can be shown that if a single ray is deleted from 33-40, the contradiction disappears." Wrong! Proof: 32-40

MMPH structure and contextuality—True-implies-false does not hold for non-KS

Proposition: Vertices 1 and 5 cannot both be assigned value '1'

"Proof":
Condition (i) does not allow this

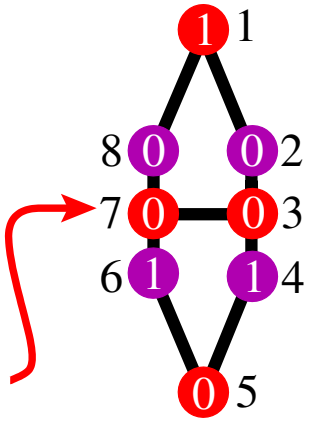
(Clifton, Horodeckis, Cabello, etc.)



8-7 $\mathcal{G}_1$ / bug/ Clifton

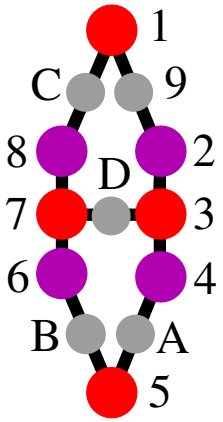
Counter-proposition: Vertex 5 cannot be assigned value '0' once vertex 1 is assigned value '1'

"Proof":
Condition (ii) does not allow this



Solution:
"Contradictions" stem from our inability to assign 0-1 values to all vertices so as to satisfy (i) and (ii).

The proposition is valid for:

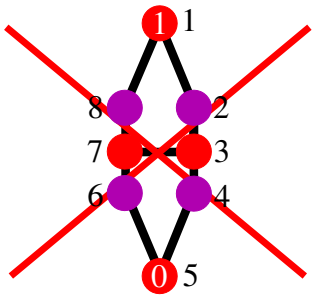


It is not contextual but it is a good show-case for the importance of predeteremined values of ...?

True-implies-false sets

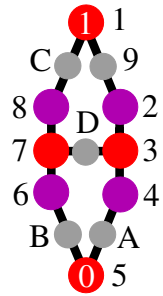
To visualise:

~~True-implies-false set~~

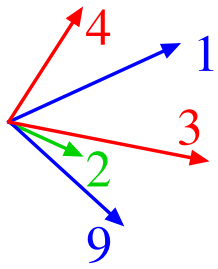


8-7 $\mathcal{G}_1$ / bug / Clifton

True-implies-false set ✓



✓



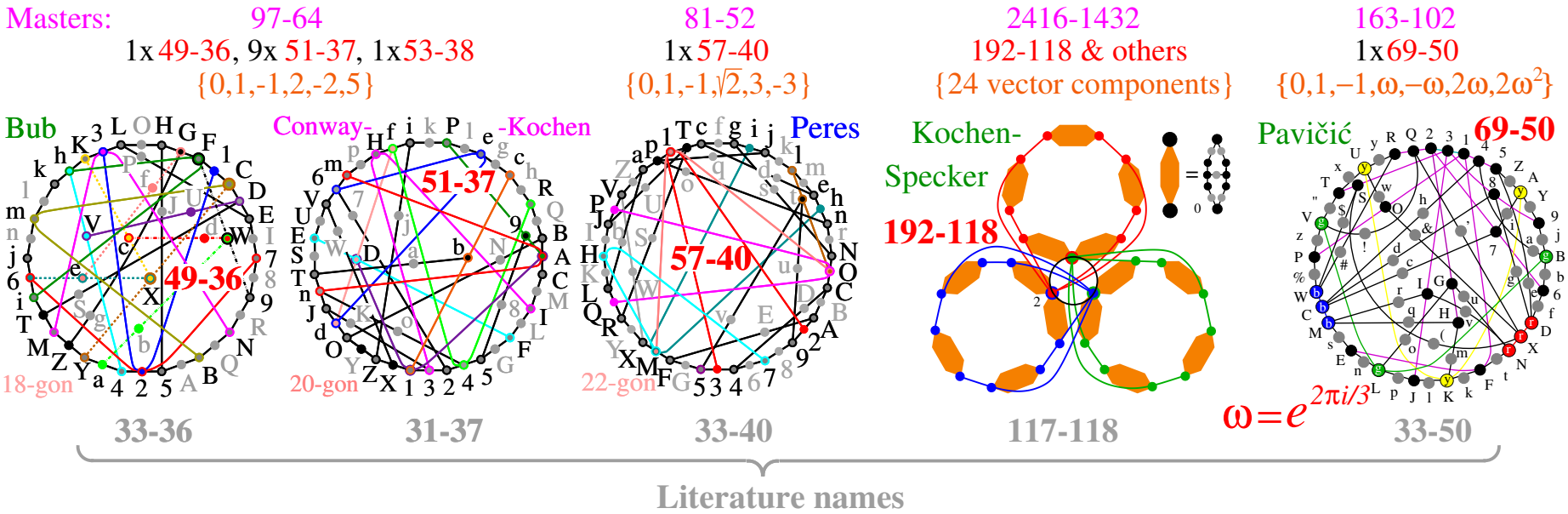
13-7

Gray vertices (multiplicity = 1) are important for geometry and/or predetermine values of vertices, but do those predetermined values or geometry or any classical model of them determine how we generate master MMPHs and therefore any MMPH? Actually, not. MMPHs might be generated by different methods none of which has anything to do with any particular classical model or hidden variable theory and geometry stems from those models.

Generation of Kochen-Specker (KS) MMPHs

Exhaustive automated generation of KS MMPHs in any dimension by methods whose complexity does not scale with dimension.

3D examples:

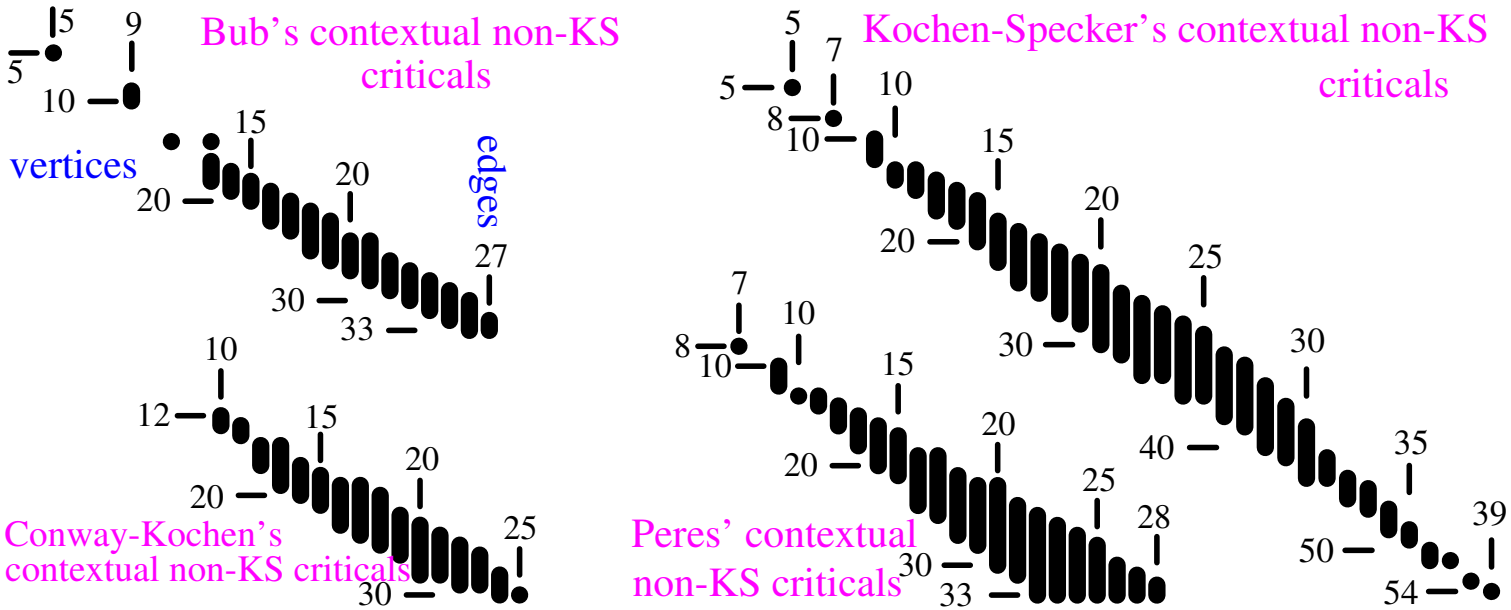


Blender 3D 69-50

https://kpavicic.com/mpavicic/HyperNav.html

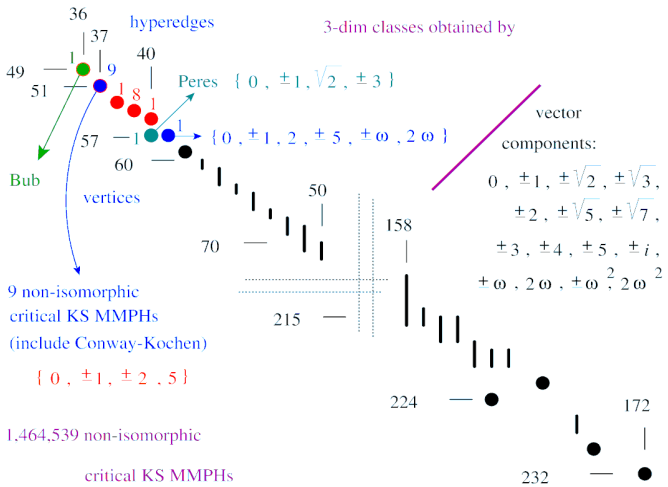
non-Kochen-Specker (non-KS) MMPHs

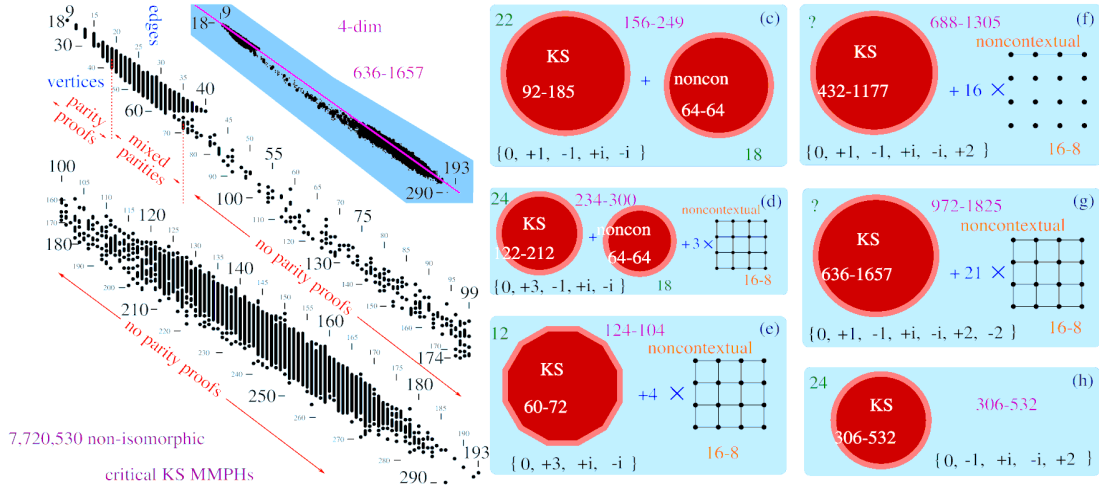
A non-binary n-dim contextual MMPH with at least one hyperedge containing less than n vertices, with a coordinatization, is called a non-KS set or a non-Kochen-Specker contextual MMPH.

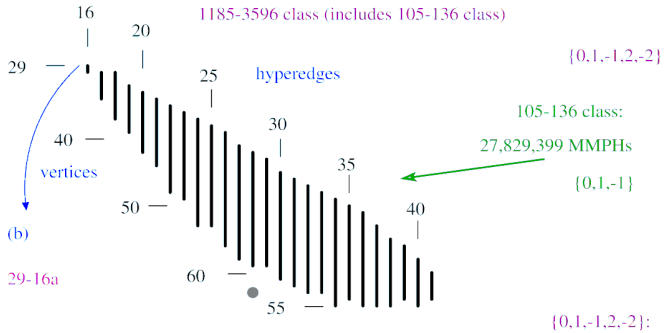


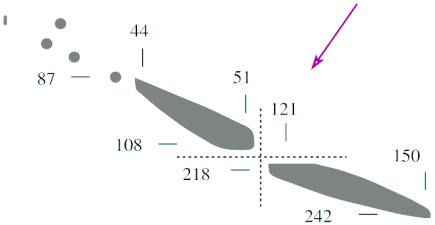
Generation of KS sets from non-KS ones is theoretically possible but not feasible because of its exponential complexity. Ergo, to call non-KS sets *"KS sets"* and KS sets *"extended KS sets"* makes no sense.

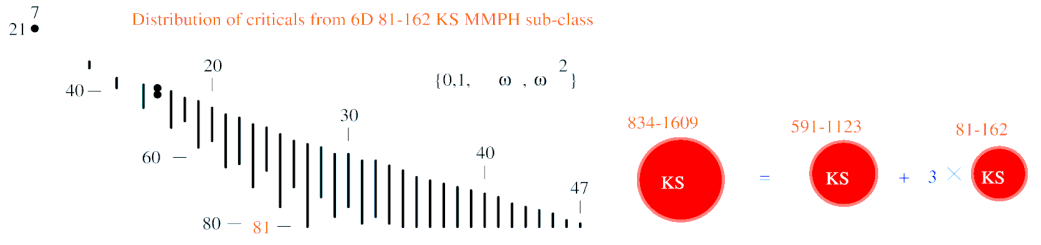
3D-6D distributions of critical KS MMPHs

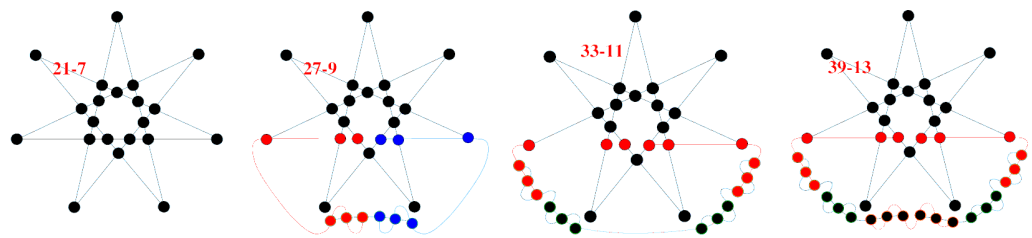




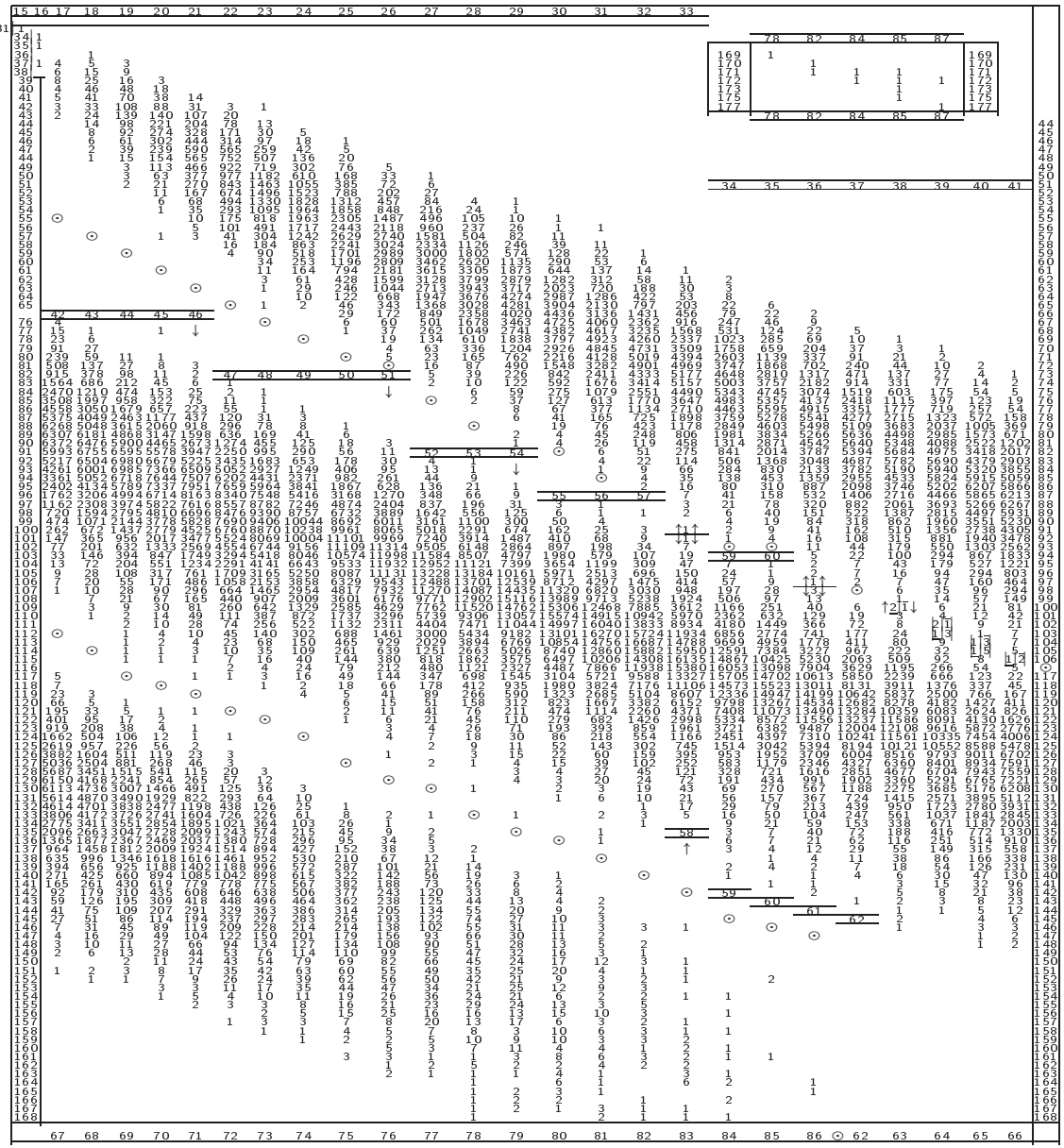






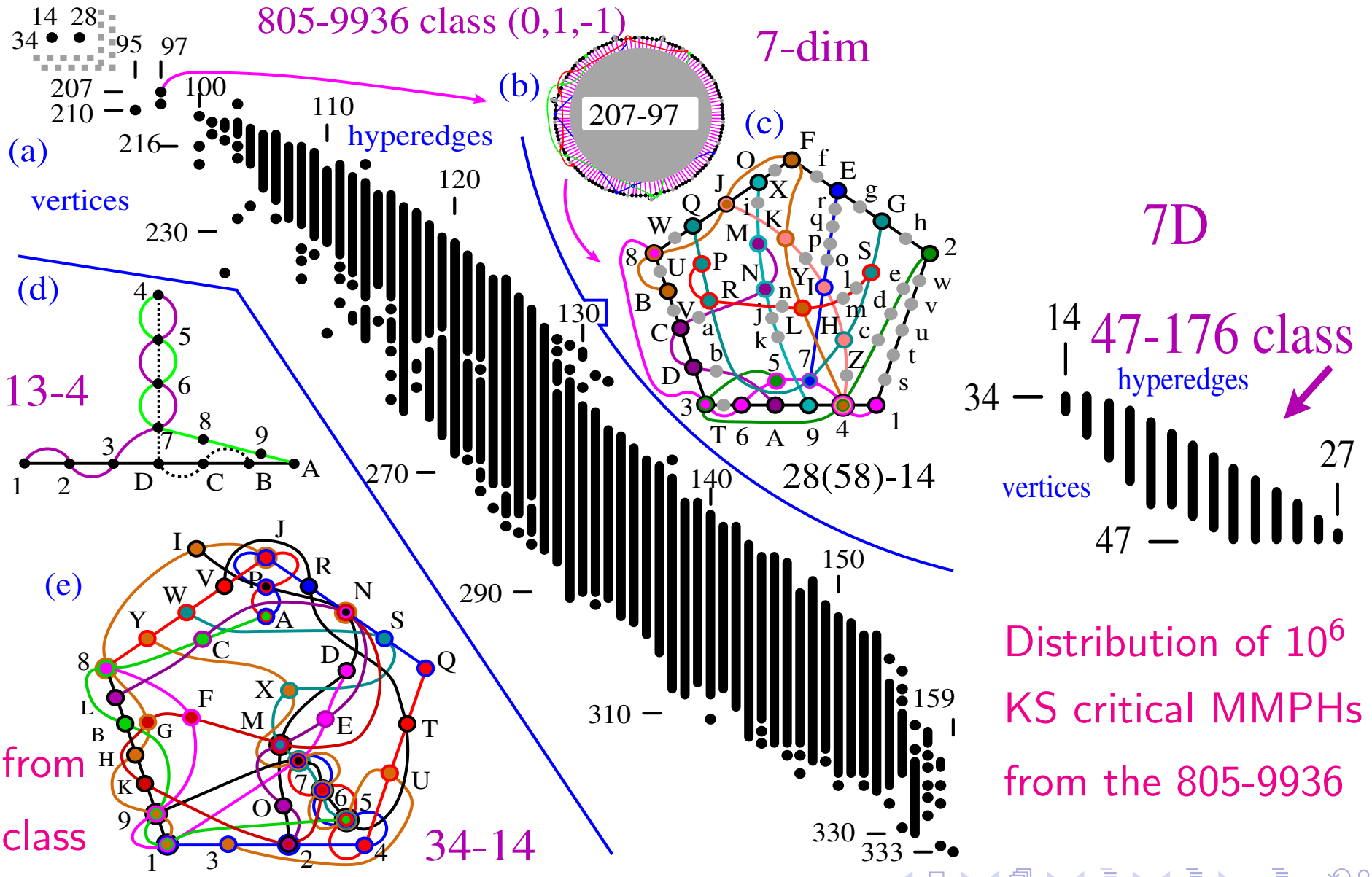


6D distributions of critical KS MMPHs from {0,1,-1}



List of 3,714,503
6D KS critical MMPHs
from the 236-1216 class

7D distributions of critical KS MMPHs from {0,1,-1}

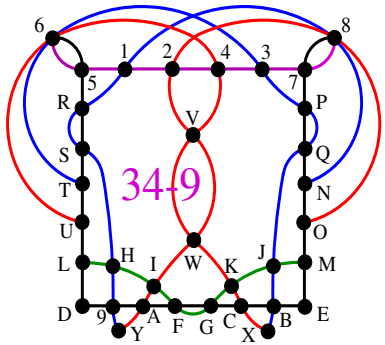


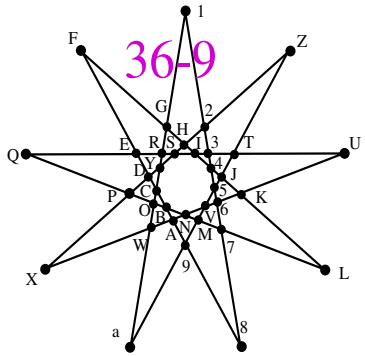
8D distributions of critical KS MMPHs from {0,1,-1}

Critical 8-dim KS sets from the 120-2024 class; 9 to 58 edges (columns) and 34 to 120 vertices (rows)

	9	11	13	15	*16*	17	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	
34	2								1																			91
36	7	⊗																										92
37		1																										93
38		7	⊗						1	2																		94
39		24	⊗							2	2																	95
40		66		⊗				1	1	6	3	3		1														96
41		86		⊗			1		1	2	6	11	5	1														97
42		128	1					1	1	5	13	23	15	5	1													98
43		45	1					1		2	11	35	43	20	7		1											99
44		42	7	⊗					1	1	15	58	113	96	34	6	2											100
45			10	⊗				1	2	1	16	64	214	315	116	29	1											101
46			9							1	11	67	299	589	451	160	22											102
47			26					1	2	2	6	62	302	958	1244	609	131	11										103
48			31					1		1	4	33	251	1180	2482	1954	695	85	4									104
49			21	3			1		1	1	4	11	178	1213	3989	4994	2525	539	49	1								105
50			16	2							2	7	94	933	4704	9535	7577	2437	291	18								106
51			4	4			1			1		6	43	625	4463	14218	17894	9153	1857	157	14							107
52			2	8	1						1	2	18	326	3840	16630	32527	25470	7987	1060	77	1						108
53				9								2	8	145	2045	15104	46228	55325	26377	5181	466	24	1					109
54				8									3	50	943	10814	50550	93880	69908	21130	2733	157	2					110
55				7		1							2	19	407	5930	42812	124765	141515	65036	12566	1107	58					111
56				3									1	6	124	2587	27584	124686	221125	154316	44413	5728	360	14				112
57				4		2							1	1	38	843	13393	93857	258485	278137	121047	22843	1978	78	1			113
58				2		3									2	229	4795	223500	58203	372836	246485	686277	8530	508	15	1		114
59						1								1		46	1352	21155	139230	361137	367153	150487	27173	2240	97	3		115
60				⊗		⊗	⊗	⊗	⊗		⊗					8	276	5978	61155	246651	383959	237596	62918	7562	418	10		116
62						1										41	8	1276	18859	115068	274469	257663	100166	17092	1361	54		117
63						1	1										6	167	3793	35175	127956	180895	103768	25390	2951	145	7	118
64							1											13	474	6262	34759	74203	62600	22408	3638	272	11	119
65							2	1											24	513	4102	13192	16530	8426	1901	211	14	120
67							1																					
68							1																					
69							1	1																				
71								1																				
72								1																				72
73								1																				73
77										1																		77
78											2																	78
79										2	1																	79
80								1		2	1	1																80
81									1			4																81
82											1	1																82
83											1	1																83
84												1	2															84
85										⊗			1															85
87												1		1														87
88												1	1	1														88
89														1	1													89
91														1														91
92																2												92
93															1	1												93
94																		1										94
95																2												95
97																	1	1										97
98																												98
99																	1	1										99
100																1												100
	9	11	13	15	*16*	17	*19*	21	*23*	25	*27*	28	29	30	31	34	35	36	37	51	52	53	54	55	56	57	58	

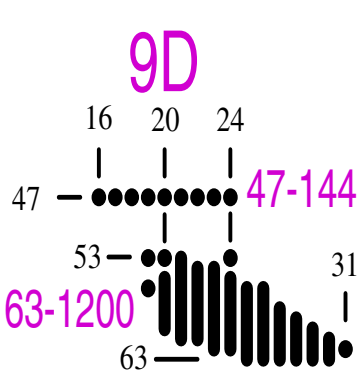
8D

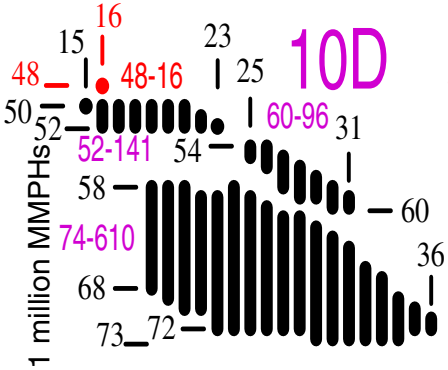


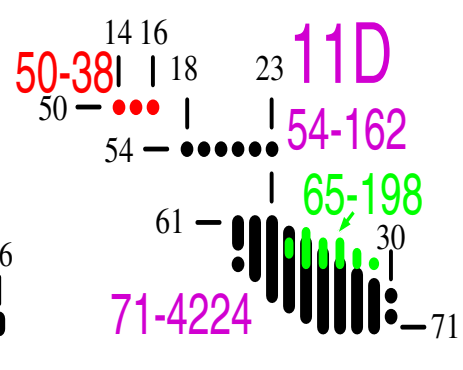


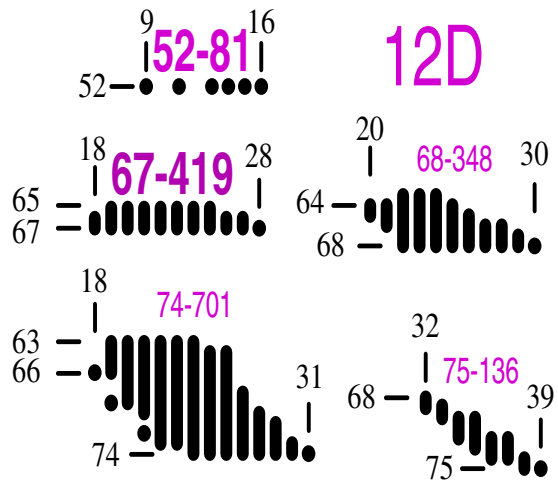
List of 6,925,540
8D KS critical MMPHs
from the 120-2024 class

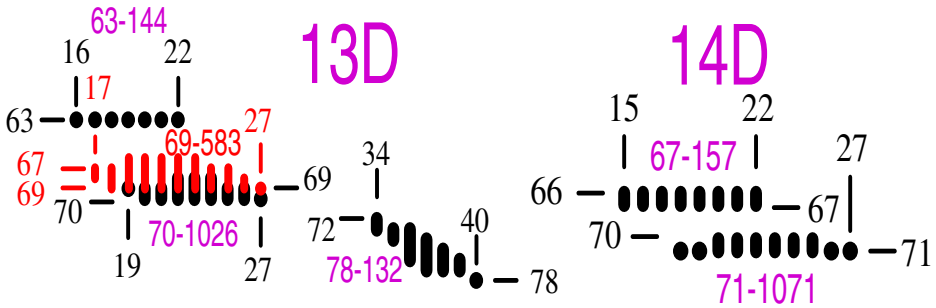
9-32D distributions of critical KS MMPHs from {0,1,-1}

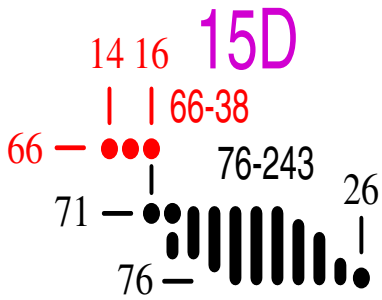


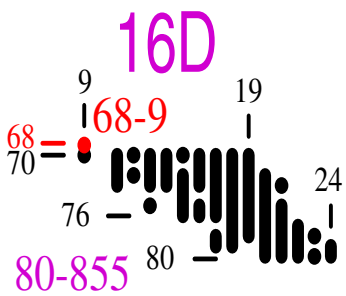


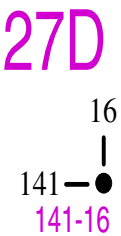












Critical 32-dim KS sets from the 160-661 class; 11 to 29 edges (columns) and 135 to 160 vertices (rows)

	11	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	
135										1									135
⋮										⋮									⋮
144	31	4																	144
145		21		2															145
146		100	15	59	11														146
147		253	54	400	219	124	10												147
148		301	61	1337	1548	1780	553	59		1									148
149		226	26	798	523	675	212	24	1										149
150		217	16	1895	2605	7129	5507	1379	53	1									150
151		127	3	976	513	1350	776	286	13	2	1								151
152		65	5	1647	1733	9609	13324	8862†	1578	60									152
153				852	380	2629	3016	2547	683	80	1		1						153
154			1	739	440	5974	11667	16824	7802	1278	67	1							154
155				283	93	2015	2998	5901	3743	945	91	1							155
156				152	66	2476	5171	14443	13922	6035	1027	66	4			1			156
157				34	8	700	976	4478	6140	4048	974	117	5	1		1			157
158				10	6	451	906	4866	9188	9879	4395	942	82	2	2	2			158
159				3	3	49	43	475	1285	2556	2269	1055	228	17	3	1	2		159
160				1		57	91	861†	2405	4272	4133	2071	562	74	5	2	2	2	160
	11	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	

32D

Features of contextual sets

(a) vertices of multiplicity 1 (belonging to just one hyperedge; gray dots) play important role in geometry and features of MMPHs;
(b) smallest critical MMPHs might serve as building blocks of all MMPHs;
(c) on the one hand, properties of MMPHs does not rely on their coordinatization (vertices need not be vectors) and on the other, small vector components generate practically all known MMPH of any complexity in any dimension;
(d) MMPHs can be generated from vector components (via programs in an automated way), polytopes, Lie algebra, Pauli operators, geometry, symmetries, . . . ;
(e) MMPHs contextuality can imply nonlocality features and vice versa;
(f) assumed predetermined values within hypothetical classical models (e.g., hidden variable ones) can be theoretically and experimentaly disproved with various tools such as noncontextuality inequalities or pseudo-telepathy protocols.

What are applications of contextuality?

What we can deduce from the way we engineer the contextual sets is that via a series of different methods we arrive at sets which turn out to be contextual or from which we filter out those ones that are contextual or which we construct in such a way to be contextual.

Contextual sets do not stem from any particular well defined classical or hidden variable theory.

In the literature, applications of contextual sets are mostly prescriptions of how to verify that they do not allow assignments of predetermined values to their vertices. Here come various inequalities, pseudo-telepathy games, quantum cryptography, and other tools of such undertakings.

Since it is obvious what such verification would yield, it seems more challenging to explore whether contextual sets can have applications within the quantum theory itself without a reference to any classical theory.

Is the contextuality we encounter when dealing with particular quantum setups random or structuraly conditioned?

Let us consider a few examples of potential quantum *oomph* applications.

Quantum Computation Magic

QUANTUM COMPUTING

Powered by magic

What gives quantum computers that extra oomph over their classical digital counterparts? An intrinsic, measurable aspect of quantum mechanics called contextuality, it now emerges. SEE ARTICLE P.351

ARTICLE 19 JUNE 2014 | VOL 510 | NATURE | 351

doi:10.1038/nature13460

Contextuality supplies the 'magic' for quantum computation

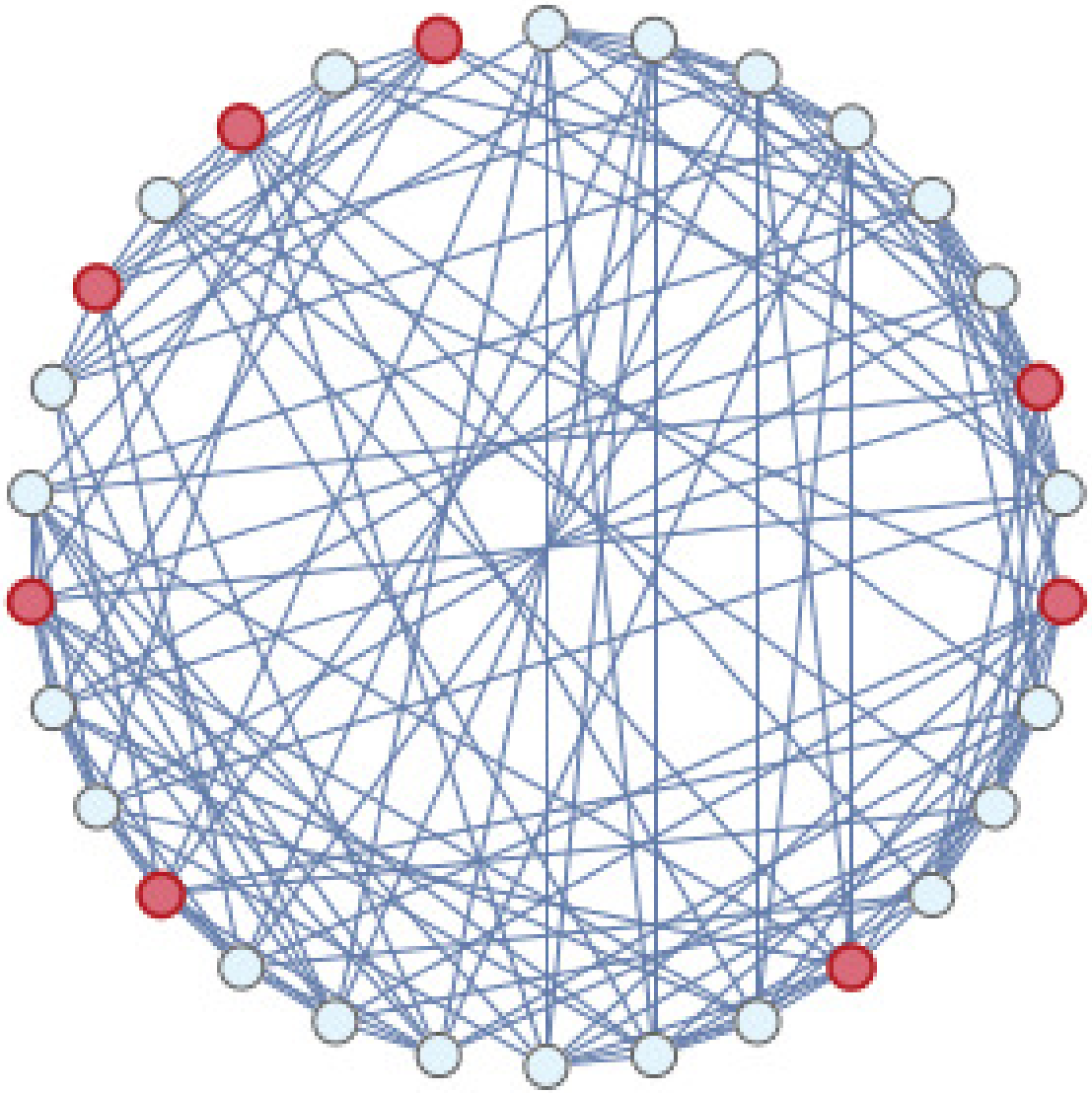
Mark Howard[1,2], Joel Wallman[2], Victor Veitch[2,3] & Joseph Emerson[2]

Quantum computation background

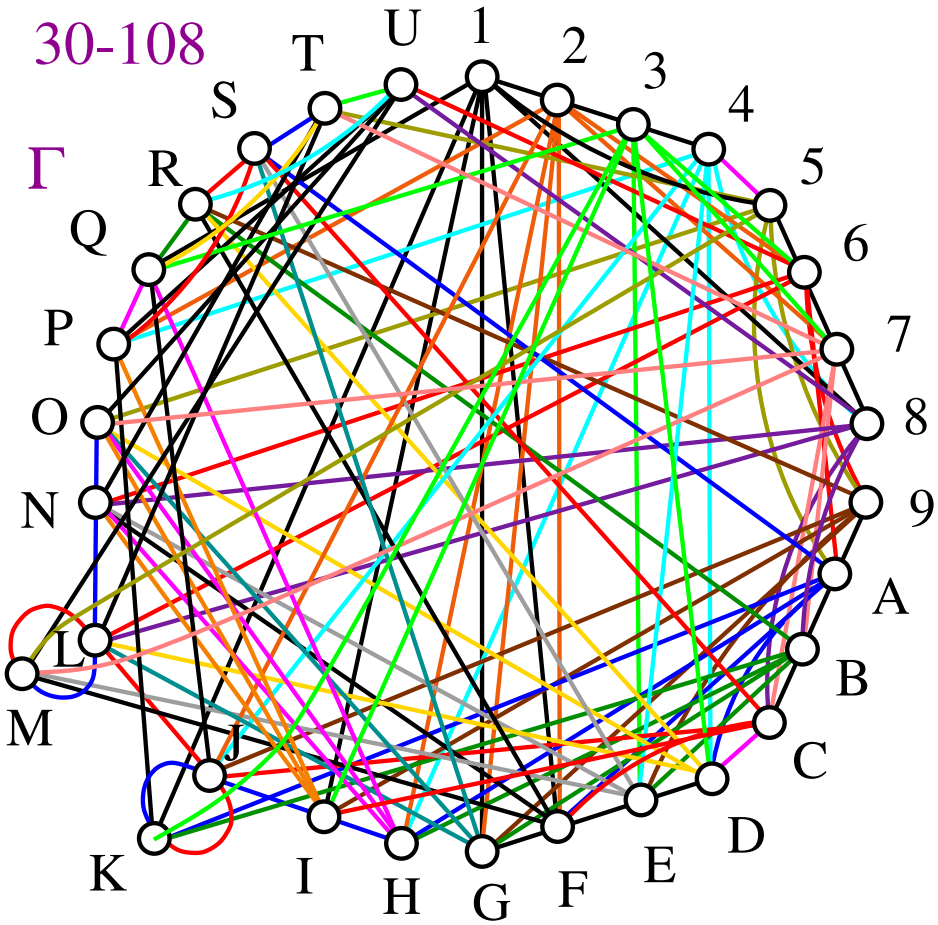
Quantum computation with qubits is most efficient when stabilizer circuits constructed via Clifford gates (Hadamard, c-NOT, and phase gate) are used. They can be simulated on a classical computer (Gottesman–Knill theorem), by when combined with ancillary qubits in magic states they yield a quantum speedup and turn out to be contextual.

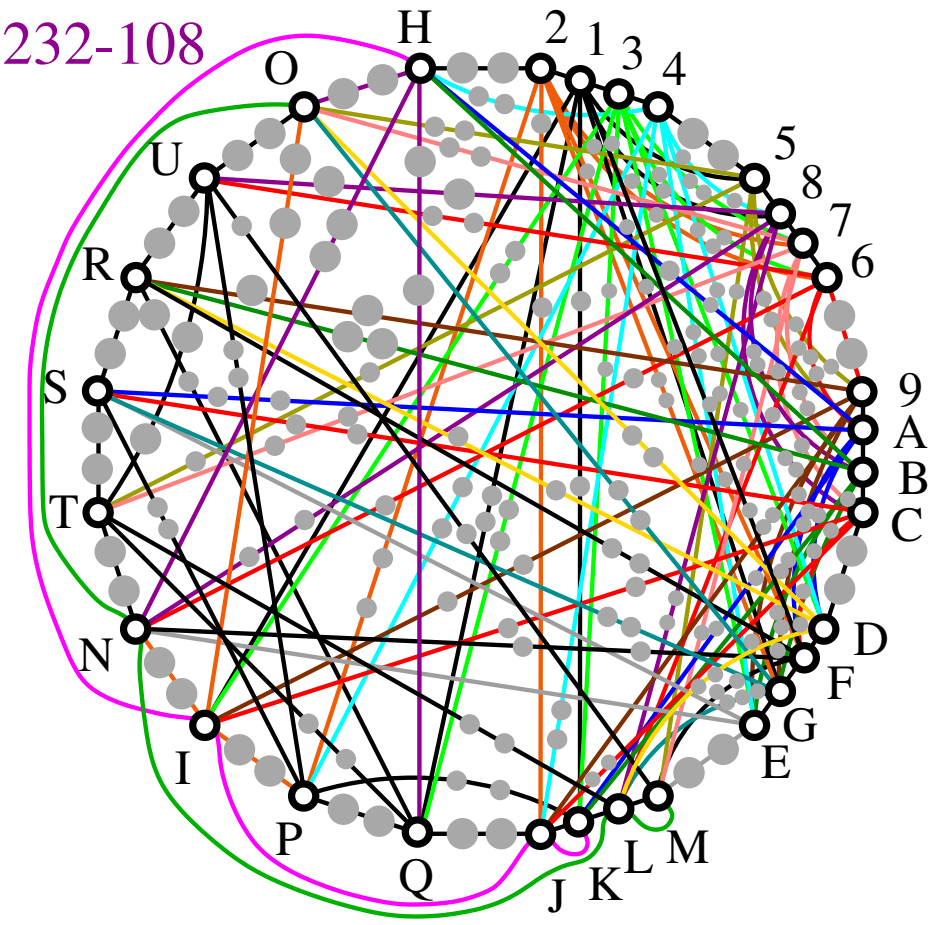
Two-qubit stabilizer states used in error-correction design yield graph Γ (30-108) shown in the next slide. Vertices correspond to the states and their orthogonalities/hyperedges correspond to the effect of ancillary qubits in magic states on them.

Maximum independent Γ set of 2-qubit stabilizer states

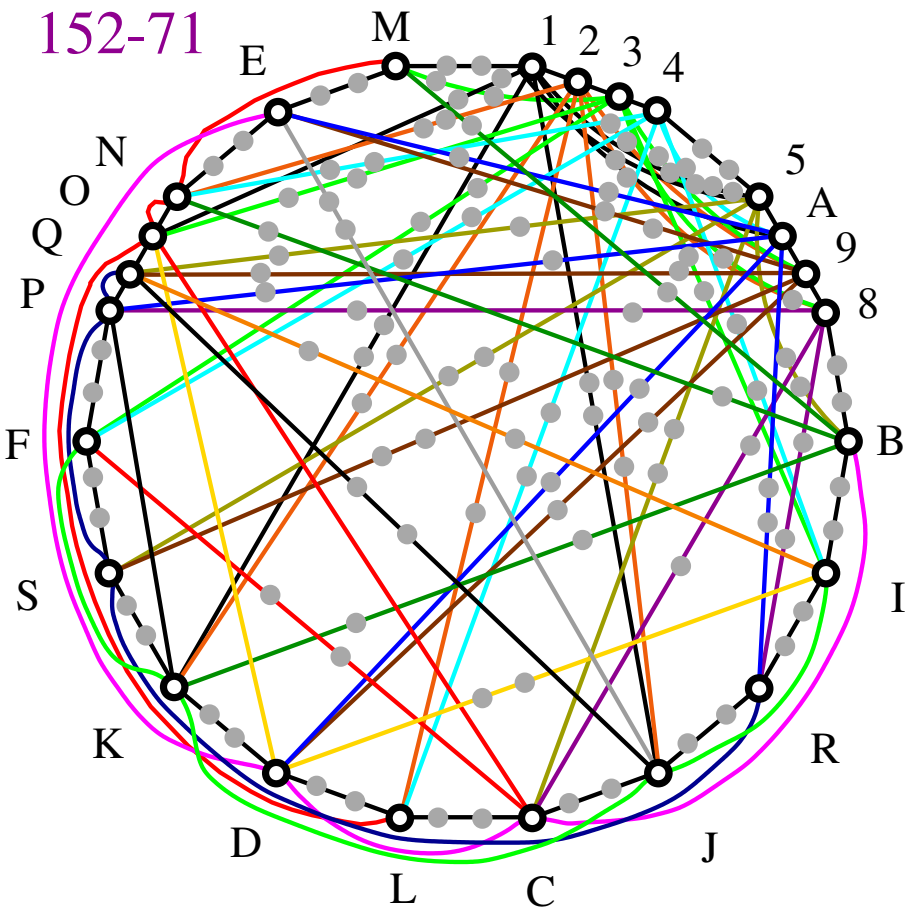


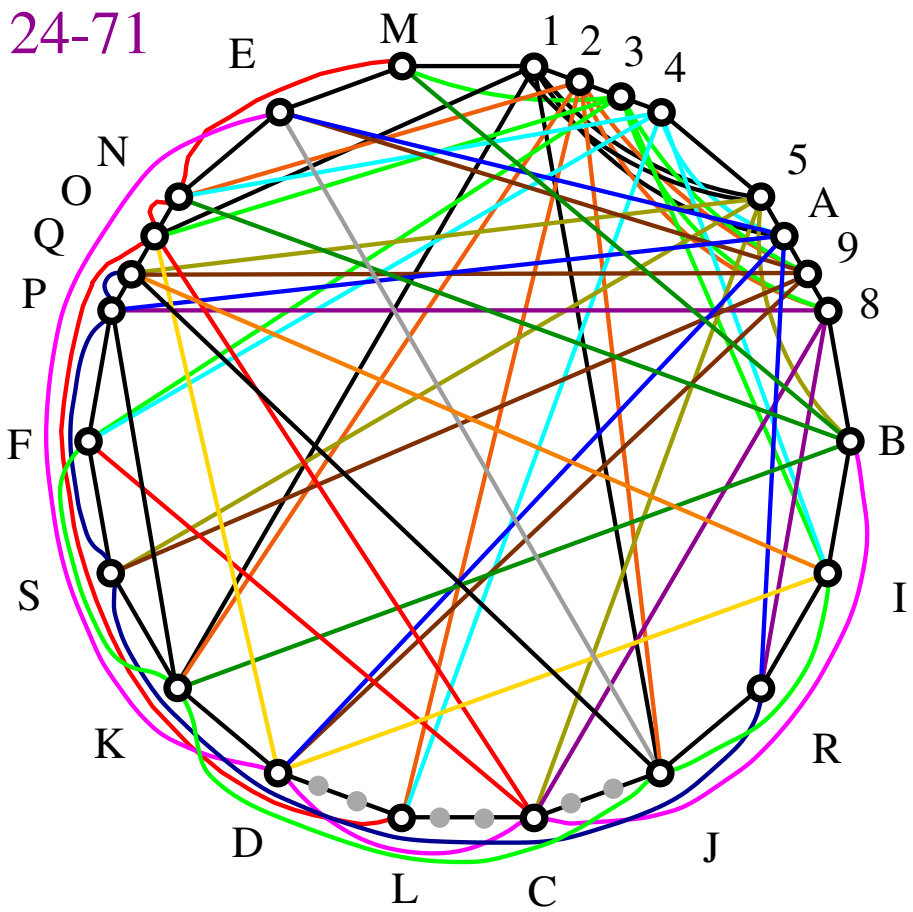
non-KS MMPH of Γ and its non-critical KS (obtained from non-KS when vertices of multiplicity 1 are added to it)





The 152-71—the only critical KS sub-MMPH of 232-108 and its non-KS subset 24-71.





Open questions

To which stabilizer states do the gray vertices correspond?

Does the critical 152-71 KS MMPH and even 24-71 non-KS MMPH represent a possible simplification of the stabilizer states and therefore also of the error correction design?

How many of the stabilizer states enhanced by the ancillary magic states are contextual? Are there noncontextual ones that can serve the same purpose?

S-Hadamard matrices

Most quantum computation algorithms are based on the Fourier transform, of which the Hadamard transform is a special case. Recently, the S class of H matrices, has been designed to prove the existence of KS hypergraphs in an n-dim space [84].

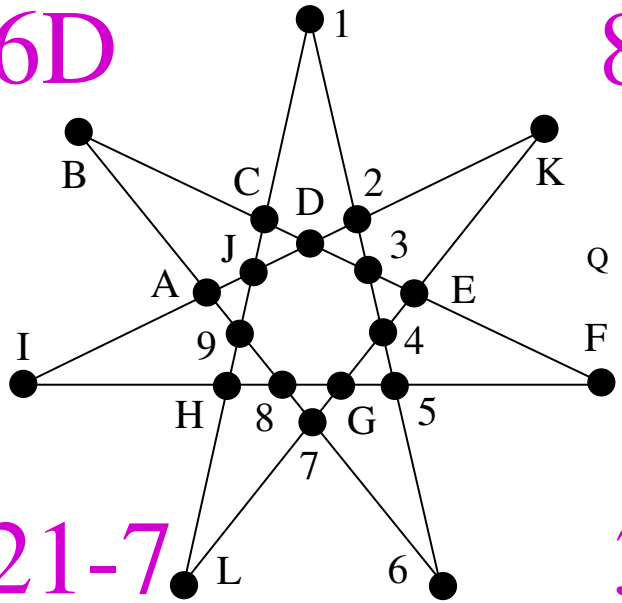
Lisoněk 2019: Suppose that there exists an S-H matrix of order n (n even); then there exists a KS hypergraph k-l in $\mathbb{C}^n$ such that $k = \binom{n+1}{2}$ and $l = n + 1$.

Let us go *beyond* existence:

KS MMPHs $\Leftrightarrow$ S-Hadamard matrices

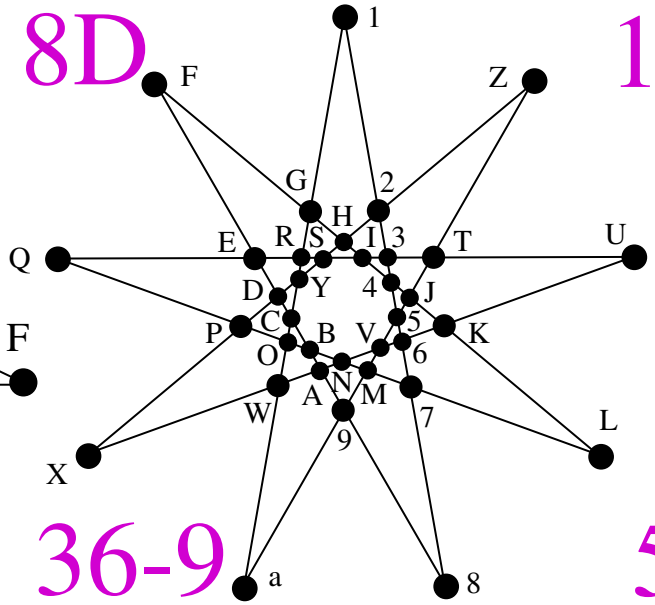
All the corresponding KS MMPHs turn out to be star-like; This clarifies why n has to be even: one cannot draw a regular star with the Schläfli symbol $\{n+1/\frac{n}{2}\}$ in odd-dimensional spaces because $\frac{n}{2}$ has to be integer.

6D



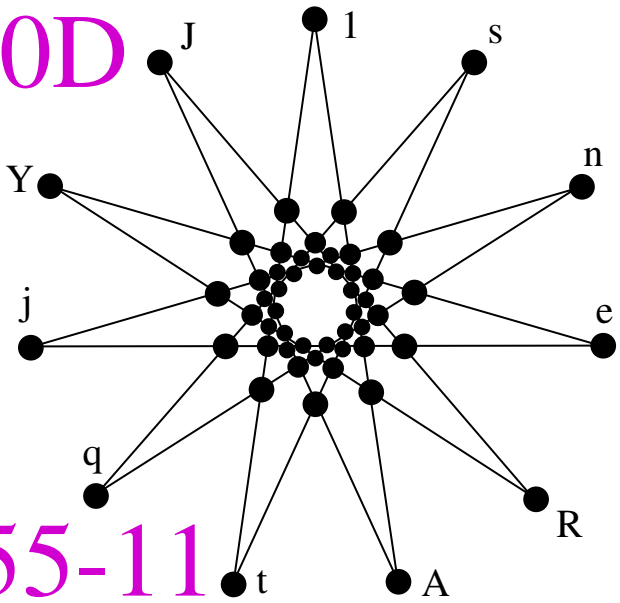
21-7

8D



36-9

10D

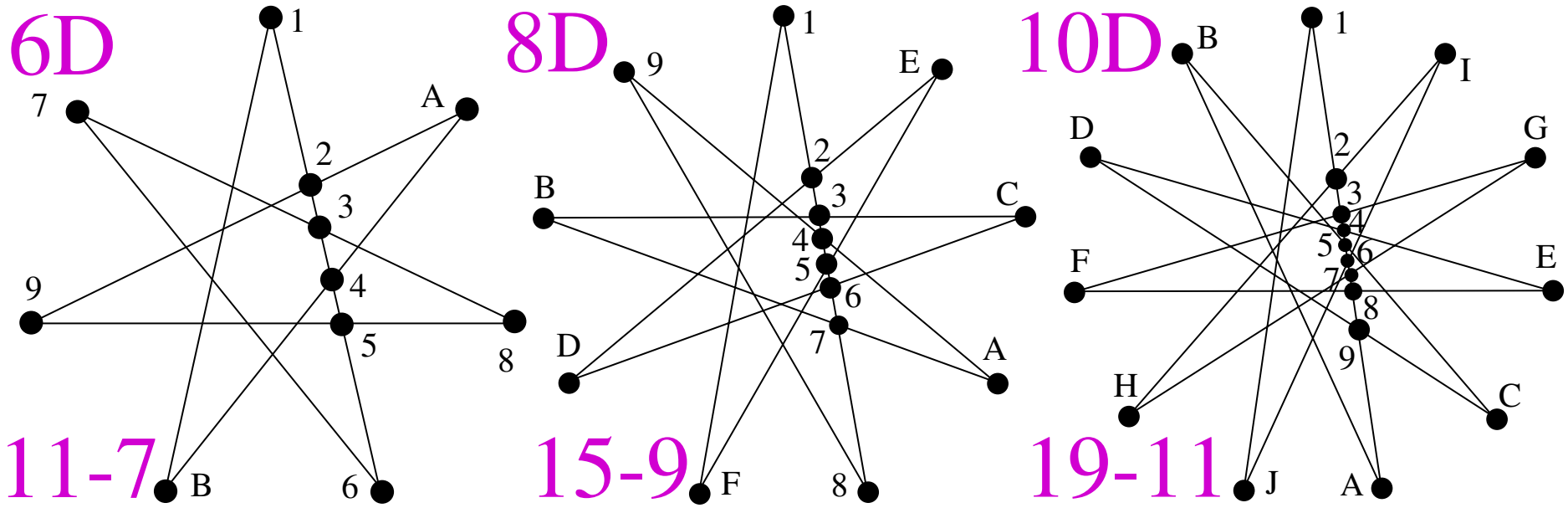


55-11

Drawback: CPU time very demanding.

non-KS critical MMPHs $\Leftrightarrow$ sub-S-Hadamard matrices?

It turns out that any contextual star-like MMPH with at least one complete hyperedge (n vertices) is critical.

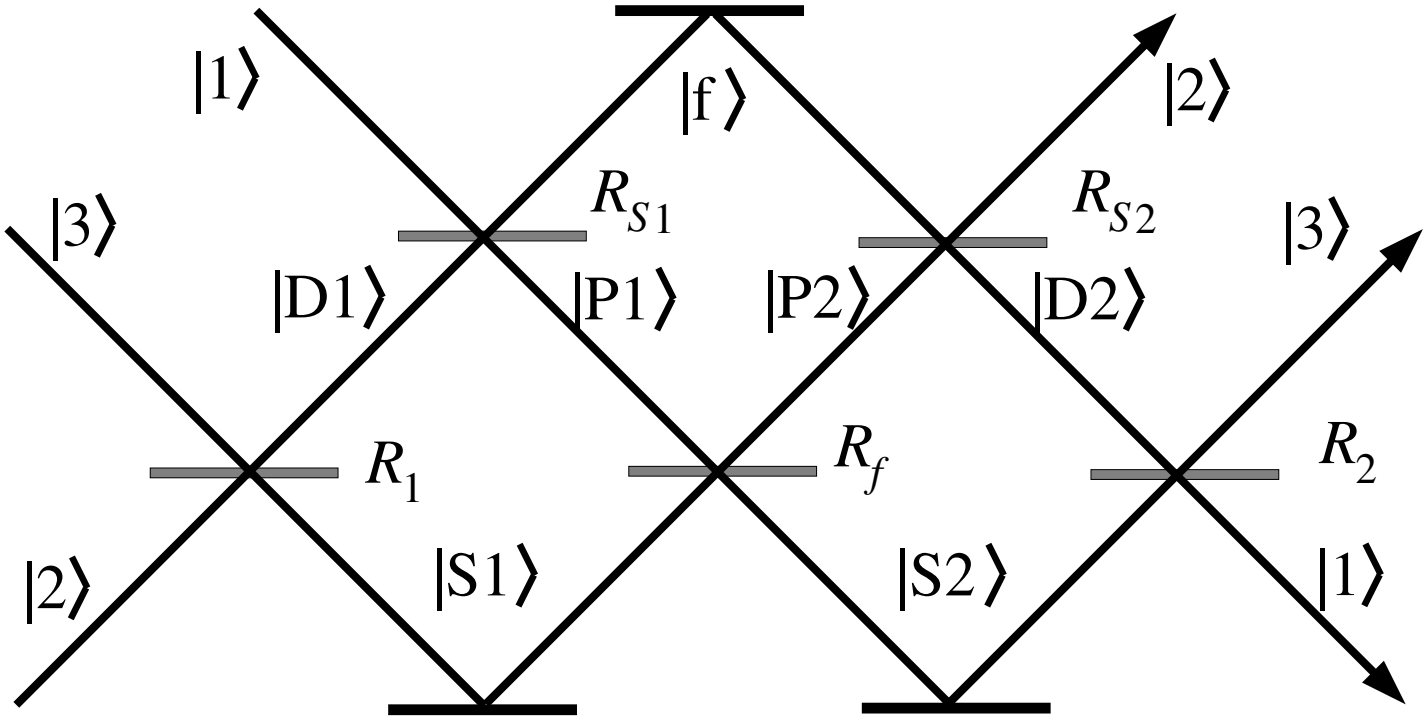


Advantage: Easy to compute.

Open question: Do they correspond to some sub-classes of S-Hadamard matrices and transforms? If so, which ones?

Hofmann's weak measurements

Hofmann's three path interferometer enables verification of Hardy-type contextuality of state $\Psi = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |11\rangle)$ via weak value measurements of path projectors.

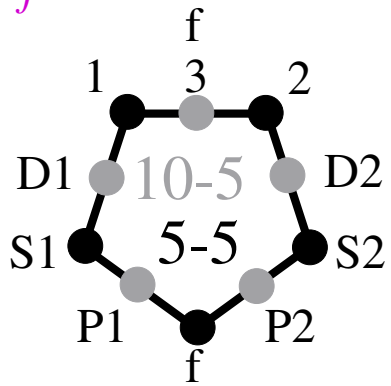


MMPH of the measurements

$R_1 = R_2 = 1/2$

$R_{S1} = R_{S2} = 1/3$

$R_f = 1/4$



Which Hofmann's extended weak interferometer measurements are contextual?

Weak measurements can be useful for implementations apart from exhibiting quantum contextuality.

Hofmann's weak interferometer measurements are scalable.

How many of them are contextual?

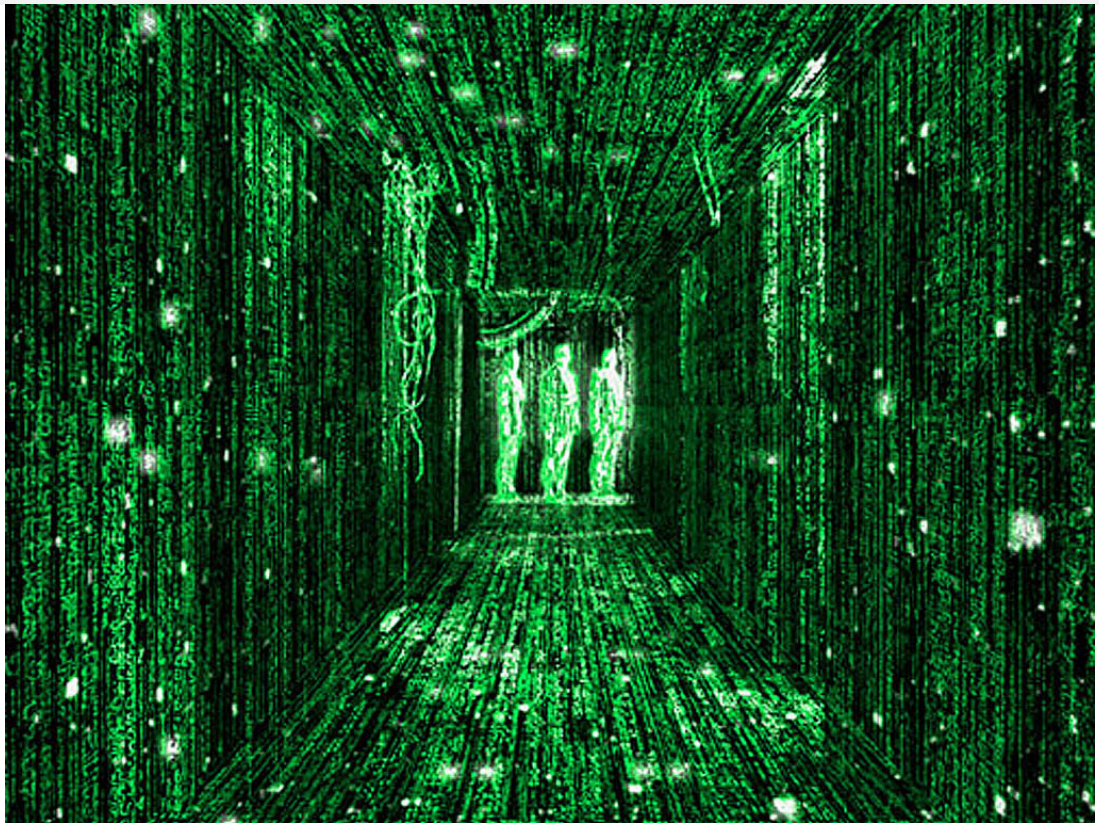
Rossi, Zjawin, Baldijão, Schmid, Selby, and Sainz: How typical is contextuality?

They posed the question as a classical-quantum dichotomy: "How typical is it to stumble upon contextuality when we randomly sample a finite number of pure states and projective measurements?"

Their answer: "Using numerical linear programs to test for the existence of a generalized-noncontextual model, we find that contextuality is fairly common: even in experiments with only a modest number of random preparations and measurements, contextuality arises with probability over 99%."

Conclusion: What seems to be the most important in the 'classical-quantum dichotomy' is the contextuality structure of quantum measurements defined, e.g., by the features of the corresponding hyperegraphs. Using programs for testing generalized-contextual model is needed.

Thanx for your attention 😎



Institute Ruđer Bošković, Zagreb, Croatia. mpavicic@irb.hr

For references, please, go to:

https://mladen-pavicic.github.io/web/